Preservation of robustness, non-fragility and passivity for controllers using linear fractional transformations

GUILLERMO FERNÁNDEZ-ANAYA
JOSÉ-JOB FLORES-GODOY
Universidad Iberoamericana
Dep. Física y Matemáticas
Prol. Paseo de la Reforma 880
Lomas de Santa Fe
México, D. F. 01210
MÉXICO
guillermo.fernandez@uia.mx
job.flores@uia.mx

JOSÉ ÁLVAREZ-RAMÍREZ
Universidad Autónoma Metropolitana
Iztapalapa
Div. Ciencias Básicas e Ingeniería
Apartado Postal 55-534
México, D. F., 09340
MÉXICO
jjar@xanum.uam.mx

BASILIO DEL MURO
Instituto Politécnico Nacional
ESIME Culhuacán
MÉXICO
bdelmuro@ipn.mx

Abstract: In this work, using algebraic methods, we characterize the parameters of a linear fractional transformation such that the composition of a class of rational function with the linear fractional transformation preserves stability, in the case that the rational function is stable, or stabilizes the original rational function, in the case that the rational function is unstable. As a consequence, we obtain a dual result about the robust stabilization of a plant—represented as a rational function—compensated with a controller when there is a nonlinear disturbance induce by function composition on the parameters of the controller. This implies the non-fragility of the controller and also the plant robust stabilization for the same class of disturbances. Also, for a particular choice of one of the parameters in the linear fractional transformation, the composition of functions preserves the structure of Proportional, Proportional-Derivative and Proportional-Derivative-Integral type of controllers. Finally, results about stabilization based in passivity using the linear fractional transformation are given.

Key–Words: Linear fractional transformation, composition, robust stabilization, PD/PI/PID controllers

1 Introduction

Recently in literature it has has appeared a series of articles on the subject of preservation of stability for linear systems in the frequency domain, [1, 3, 4, 5, 8]. In [5] it is presented a method on maps that preserve the stability of stable polynomials, i.e., the map that is obtained by multiplying the vector of coefficients of stable polynomials by a fixed matrix to obtain a vector of stable coefficients. This method does not have a complete characterization of which matrices preserve stability. Other methods, used a substitution of a rational function in a polynomial to guarantee stability and are based on H-domains and diagrams of Mikhailov [8]. In [1], it is used the substitution $\alpha(s) = \frac{as+b}{cs+d}$ for the s variable in a stable rational function and it is proven that for positive real numbers a, b, c and d, such that $ad - bc \neq 0$, this substitution preserves stability, but the case is very restrictive. In [3, 4], the results from [1] are extend and generalize, showing that substitutions of the s variable in a rational stable function by a strictly positive real functions with relative degree zero, preserve stability, and under some additional conditions, powers of functions SPR0, also preserve stability, but only sufficient conditions are given. In this work using algebraic methods, we give a complete characterization on the parameters of a linear fractional transformation, $\alpha(s) = \frac{as+b}{cs+d}$, such that the composition of a class of rational, real, proper, stable or unstable functions, $H(s) = \frac{N_h(s)}{D_h(s)}$, with the linear fraction transformation $\alpha(s)$ is stable, i.e., find the parameters a, b, c and d such that $H(\alpha(s))$ is stable, with $\alpha(s) = \frac{as+b}{cs+d}$. These results generalize and extend previous results [1, 2, 3, 4]. In addition, it is possible to answer the open problem proposed in [5] for the case of maps that preserve stability, obtained under the substitution of the variable s by $\alpha(s)$ in a stable polynomial. This is done by characterizing all the maps obtained under this substitution, that preserve stability for any stable polynomial $D_h(s)$ which is mapping to stable polynomial $(cs+d)^m D_h(\alpha(s))$. In other words, we characterized the space of parameters a, b, c and d for which the map, preserves stability for any stable polynomial, mapping stable polynomials in stable polynomials. But also we characterized the space of parameters a, b, c and d for which the map, stabilizes unstable polynomials. As a consequence, we obtain a dual result, in the sense that the robust stabilization of a plant $H(s)$ with disturbances induced by the substitution of the variable s by $\alpha^{-1}(s)$, with a controller $C(s)$, implies the non-fragility of the controller $C(s)$ under the same class of disturbances, induced by the substitution of the variable s by $\alpha(s)$, in the controller, and vice versa i.e., the non-fragility of the controller $C(s)$ under disturbances induced by the substitution of the variable s by $\alpha^{-1}(s)$, implies the robust stabilization of a plant $H(s)$ with disturbances, induced by the substitution of the variable s by $\alpha(s)$ with a controller $C(s)$. In the particular case when $b = 0$, the substitution of the s variable by $\alpha(s)$, preserves the structure for Proportional-Derivative (PD), Proportional-Integral (PI) and Proportional-Integral-Derivative (PID) controllers. Based on the resulting pseudo-parametrization for these controllers class after the substitution of the variable s by $\alpha(s)$, taking $b = 0$, we mention some ideas that could be used later for tuning rules on the derivative part for the PD controllers, and for the proportional and integral parts of the PI controllers. Finally, we give two results about stabilization based on passivity.

2 Preliminaries

This section we give the necessaries definitions and notation used though out the paper.

Let $\mathbf{R}(s)$ be the set of rational functions with real coefficients. Consider a rational function $H(s) \in \mathbf{R}(s)$

$$H(s) = \frac{N_h(s)}{D_h(s)} = k\frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0}$$

where $N_h(s)$ and $D_h(s)$ are coprime, with $m \geq n$. Let us factorize $H(s)$ as $H(s) = H_r(s)H_c(s)$, where

$$H_r(s) = k\frac{(s - z_1)\cdots(s - z_{n-l_1})}{(s - p_1)\cdots(s - p_{m-j_1})}$$

has real poles and zeros, $l_1 < n$, $j_1 < m$; and

$$H_c(s) = \frac{\left[(s-\rho_1)^2 + \nu_1^2\right]\cdots\left[(s-\rho_{l_0})^2 + \nu_{l_0}^2\right]}{\left[(s-\sigma_1)^2 + \omega_1^2\right]\cdots\left[(s-\sigma_{j_0})^2 + \omega_{j_0}^2\right]}$$

has complex poles and zeros, $l_0 = \frac{l_1}{2}$ and $j_0 = \frac{j_1}{2}$.

Definition 1 ([9]) *A function $G(s)$ (rational or irrational) of complex variable $s = \sigma + \mathrm{j}\omega$ ($\mathrm{j} = \sqrt{-1}$) is positive real (PR) if $G(s)$ is analytic in $\mathrm{Re}[s] > 0$ (stable), $G(s)$ is real for s real and $\mathrm{Re}[G(s)] \geq 0$ for all $\mathrm{Re}\, s > 0$.*

Definition 2 ([6, 7]) *A real and rational function $H(s)$ is strictly positive real (SPR) if $H(s)$ is analytic in $\mathrm{Re}[s] \geq 0$ and $\mathrm{Re}[H(\mathrm{j}\omega)] > 0$ for all $\omega \in R$. Moreover, a real and rational function $p(s)$ is $\mathrm{SPR0}$ if it is SPR and has zero relative degree.*

Let us define the following sets:

$$\mathrm{SPR0}^* = \Big\{p(s) \in \mathbf{R}(s) : p(s) \text{ is } \mathrm{SPR0}\Big\} \cup \{s\}.$$

$$\Gamma(a,b,c,d) = \Big\{\alpha(s) \in \mathbf{R}(s) : \alpha(s) = \frac{as+b}{cs+d}, \quad ad - bc \neq 0 \text{ and } a,b,c,d > 0\Big\} \cup \{s\}.$$

The following properties can be easily verified for the set $\Gamma(a,b,c,d)$

1. $\lim_{(b,c)\to(0,0)} \frac{as+b}{cs+a} = s$, where $a^2 - bc > 0$;
2. if $\alpha(s),\ \beta(s) \in \Gamma(a,b,c,d)$, then $\alpha(\beta(s)),\ \beta(\alpha(s)) \in \Gamma(a,b,c,d)$.

From the associative property of function composition, we know that the set $\Gamma(a,b,c,d)$ is a non commutative monoid under the composition operation. Additionally, is well know that $\Gamma(a,b,c,d) \subset \mathrm{SPR0}^*$, [1].

3 On the preservation of stabilization, fragility and passivity in PI, PD and PID controllers

Consider a Single-Iput Single-Output (SISO) Linear Time-Invariant (LTI) system with state variable representation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}$; and transfer function representation

$$H(s) = C\,(sI - A)^{-1} + D$$

We use a linear fraction transformation, $\alpha(s) = \frac{as+b}{cs+d}$, to obtain the SISO LTI system $H(\alpha(s))$ with state variables representation

$$\begin{aligned} \dot{x} &= (dA - bI)\,Q_A x + (ad - bc)\,Q_A B u \\ y &= CQ_A x + (D + cCQ_A B)\,u \end{aligned} \tag{2}$$

with $Q_A = (aI - cA)^{-1}$.

The problems to study in this section are the following:

1. If system (1) is stable, for which set of parameters a, b, c, d, system (2) is stable?

2. If system (1) is unstable, for which set of parameters a, b, c, d, system (2) is stable?

The answers to the previous questions are given in the following two results.

Lemma 3 *Consider a plant* $H(s) = \frac{N_h(s)}{D_h(s)}$ *where* $N_h(s)$ *and* $D_h(s)$ *are polynomials satisfying* $\deg D_h(s) = m \geq \deg N_h(s) = n$. *Let us also define the lineal fractional transformation* $\alpha(s) = \frac{as+b}{cs+d}$ *where a, b, c and d are real numbers such that* $cd \neq 0$ *and* $ad - bc \neq 0$.*Let us substitute the s variable by* $\alpha(s)$ *in* $H(s)$, *i.e.,* $H(\alpha(s))$. *Then,* $H_\alpha(s) \equiv H(\alpha(s))$ *is stable if and only if the following conditions holds:*

1. *Either* $p_i d - b > 0$ *and* $a - p_i c < 0$, *or* $p_i d - b < 0$ *and* $a - p_i c > 0$ *for each* $i = 1, \ldots, m - j_1$ *where* $p_1, \ldots, p_{m-j_1}$ *are the real poles of* $H(s)$*;*

2. *the parameters a, b, c and d satisfies*

$$\sigma_i^2 - \left(\frac{a}{c} + \frac{b}{d}\right)\sigma_i + \frac{ab}{cd} + \omega_i^2 > 0$$

for $i = 1, \ldots, j_0$, *where* $\sigma_i + \mathrm{j}\omega_i$ *are the complex poles of* $H(s)$.

Proof: The proof is in Appendix A.

Remark 4 *Notice that if we consider* $x = \frac{a}{c}$ *and* $y = \frac{b}{d}$, *the function* $f(x, y) = \sigma_j^2 - (x + y)\sigma_j + xy + \omega_j^2$ *has a local minimum at* $x = y = \sigma_j$, *but* $f(x, y)$ *does not have a global minimum for* $x, y \in \mathbb{R}$.

Note that the parameters a, b, c and d can be negative, and c and d must be different from zero. Moreover, no assumption on the sign of the poles of $H(s)$ is made.

The case when one or two of the parameters a, b, c and d are zero or negative is considered in the following result. It is clear that there exist only two cases that make sense for two parameters equal to zero and none for more than two parameters equal to zero.

Lemma 5 *Consider* $H(s) = \frac{N_h(s)}{D_h(s)}$ *as defined in Lemma 3, but stable, (i.e.,* $p_1, \ldots, p_{m-j_1} < 0$, $\sigma_1, \ldots, \sigma_{j_0} < 0$*). Then* $H(\alpha(s))$ *is a stable plant if* $\alpha(s)$ *and* $H(s)$ *satisfies at least one of the following conditions:*

1. $a, b, c, d > 0$ *and* $ad - bc \neq 0$, *or* $a, b, c, d < 0$ *and* $ad - bc \neq 0$*;*

2. $b, c, d > 0$, $a = 0$ *or* $b, c, d < 0$, $a = 0$*; and all the poles in* $H(s)$ *must be complex.*

3. $a, c, d > 0$, $b = 0$ *or* $a, c, d < 0$, $b = 0$*;*

4. $a, b, d > 0$, $c = 0$*;*

5. $a, b, c > 0$, $d = 0$, *and* $\max\{\sigma_1, \ldots, \sigma_{j_0}\} < \frac{a}{c}$*;*

6. $a, d > 0$, $b = c = 0$ *or* $a, d < 0$, $b = c = 0$*;*

7. $b, c > 0$, $a = d = 0$ *or* $b, c < 0$, $a = d = 0$*; and all the poles in* $H(s)$ *must be real.*

8. $a, b > 0$, $d < 0$, $c = 0$ *and* $p_i d - b < 0$ *for* $i = 1, \ldots, m - j_1$ *and* $\max\{\sigma_1, \ldots, \sigma_{j_0}\} < \frac{b}{d}$*;*

9. $a, b < 0$, $d < 0$, $c = 0$*;*

10. $a < 0$, $b > 0$, $d < 0$, $c = 0$ *and* $p_i d - b > 0$ *for* $i = 1, \ldots, m - j_1$ *and* $b - d\sigma_j > 0$ *for* $j = 1, \ldots, j_0$*;*

11. $a, b > 0$, $c < 0$, $d = 0$, $a - p_i c > 0$ *for* $i = 1, \ldots, m - j_1$ *and* $a - \sigma_j c > 0$ *for* $j = 1, \ldots, j_0$*;*

12. $a, b < 0$, $c < 0$, $d = 0$*;* $a, b, c > 0$, $d = 0$.

13. $a > 0$, $b < 0$, $c < 0$, $d = 0$ *and* $a - p_i c < 0$ *for* $i = 1, \ldots, m - j_1$ *and* $a - c\sigma_j > 0$ *for* $j = 1, \ldots, j_0$.

Proof: The proof is in Appendix A.

Remark 6 *When* $a > 0$, $b < 0$, $d < 0$, $c = 0$ *or* $a < 0$, $b > 0$, $c < 0$, $d = 0$ *then stability is not guarantee, and stable plants are not mapped into stable plants, unless* $p_1, \ldots, p_{m-j_1} > 0$ *and* $\sigma_1, \ldots, \sigma_{j_0} > 0$.

Lemma 7 *In the case when* $p_1, \ldots, p_{m-j_1} > 0$ *and* $\sigma_1, \ldots, \sigma_{j_0} > 0$ *the plant* $H(\alpha(s))$ *is a stable if at least one of the following conditions holds:*

1. $a > 0$, $b < 0$, $d < 0$, $c = 0$ *and* $p_i d - b < 0$ *for* $i = 1, \ldots, m - j_1$ *and* $b - d\sigma_j > 0$ *for* $j = 1, \ldots, j_0$*;*

2. $a < 0$, $b > 0$, $c < 0$, $d = 0$ *and* $a - p_i c < 0$ *for* $i = 1, \ldots, m - j_1$ *and* $a - c\sigma_j > 0$ *for* $j = 1, \ldots, j_0$.

Proof: The proof is in Appendix A.

Now we are going to present some applications of the former technical results to the duality between robust stabilization and fragility of controllers.

Proposition 8 *Let us consider the proper plant $H(s) = \frac{N_h(s)}{D_h(s)}$ and the proper controller $C(s) = \frac{N_c(s)}{D_c(s)}$ such that it stabilizes the plant, where $N_h(s)$, $N_c(s)$, $D_c(s)$ and $D_h(s)$ are polynomials with* $\deg D(s) = n \geq \deg N(s)$. *Also consider the linear transformation $\alpha(s) = \frac{as+b}{cs+d}$ were $a, b, c, d \in \mathbb{R}$, and let us substitute the s variable by $\alpha^{-1}(s) = \frac{b-ds}{cs-a}$ in $H(s)$. Then:*

1. *the controllers of the form $C_\alpha(s) \equiv C(\alpha(s))$ stabilizes $H(s)$, if $C(s)$ stabilizes in a robust way the plant $H_{\alpha^{-1}}(s) \equiv H(\alpha^{-1}(s))$, where the a, b, c, d parameters satisfy at least one of the conditions of Lemma 5, or the conditions of Lemma 3 for the closed loop plant:*

$$\bar{P}(s) = \frac{C(s)H_{\alpha^{-1}}(s)}{1 + C(s)H_{\alpha^{-1}}(s)}.$$

2. *the controller $C(s)$ stabilizes in a robust way the plants $H_\alpha(s) \equiv H(\alpha(s))$, if the controllers $C(\alpha^{-1}(s))$ stabilizes the plant $H(s)$, where the a, b, c, d parameters satisfies at least one of the conditions of Lemma 5, or the conditions of Lemma 3 for the closed loop plant:*

$$\hat{P}(s) = \frac{C(\alpha^{-1}(s))H(s)}{1 + C(\alpha^{-1}(s))H(s)}.$$

Proof:

1. Let us suppose that the proper controller $C(s)$, stabilizes in a robust way the plant $H_{\alpha^{-1}}(s)$, where the a, b, c and d parameters satisfies at least one of the conditions of Lemma 5, or the conditions of Lemma 3 for the stable plant $\bar{P}(s)$. From Lemma 3 or Lemma 5 and from the fact that the substitution of s by $\alpha(s)$ preserves sums, multiplication, division and inversion of rational real proper and constant functions, the plant $\bar{P}(\alpha(s))$ is then stable, where $H_{\alpha^{-1}}(\alpha(s))$ is the plant $H(s)$, and the parameters a, b, c, d are as stated above.

2. The proof of is similar to the previous item.

□

Now, for the particular case, when the controller is a PD, PI,or PID is studied.

Let us consider a PD controller of the form $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$, and a PI controller of the form $C_{PI}(s) = K_p + \frac{K_I}{s}$. These controllers can be rewritten as: $C_{PD}(s) = \frac{(K_p+K_D)s+K_p r}{s+r}$ and $C_{PI}(s) = \frac{K_p s+K_I}{s}$. Note that $C_{PD}(s) \in \Gamma(a, b, c, d)$ if K_p, K_D, $r > 0$. We can now attack the problem of the non-fragile stabilization under non linear disturbances induced by the substitution of the s variable by the linear fractional transformation $\alpha(s)$.

We then have the following results:

Corollary 9

1. *If the controller $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$ robustly stabilizes the plants $H_{\alpha^{-1}}(s)$, where the parameters a, b, c, d satisfies at least one of the conditions of Lemma 5 for the closed loop system formed by $C_{PD}(s)$ and $H_{\alpha^{-1}}(s)$, then the controllers*

$$C_{PD}(\alpha(s)) = \left(\frac{(K_p + K_D)a + K_p rc}{a + rc}\right) \frac{s + \frac{b+ld}{a+lc}}{s + \frac{b+rd}{a+rc}}$$

stabilizes $H(s)$ with $l = \frac{K_p r}{K_p+K_D}$. If $\alpha(s) \in \Gamma(a, b, c, d)$, and $K_p, K_D, r > 0$, then $C_{PD}(\alpha(s)) \in \Gamma(a, b, c, d)$.

2. *If the controller $C_{PI}(s) = K_p + \frac{K_I}{s}$ robustly stabilizes the plants $H_{\alpha^{-1}}(s)$, where the parameters a, b, c, d satisfies at least one of the conditions of Lemma 5 for the closed loop system formed by $C_{PI}(s)$ and $H_{\alpha^{-1}}(s)$, then the controllers*

$$C_{PI}(\alpha(s)) = \left(\frac{K_p a + K_I c}{a}\right) \frac{s + \frac{K_p b+K_I d}{K_p a+K_I c}}{s + \frac{b}{a}}$$

stabilizes $H(s)$. If $\alpha(s) \in \Gamma(a, b, c, d)$ and $K_p, K_I > 0$, then $C_{PI}(\alpha(s)) \in \Gamma(a, b, c, d)$.

Proof: It follows directly from Proposition 8 and from the definition and properties of the set $\Gamma(a, b, c, d)$. □

Note that the controllers $C_{PD}(\alpha(s))$ and $C_{PI}(\alpha(s))$ in Corollary 9 are not PD or PI controllers (unless $b = 0$). Both controllers are lead-lag networks. As $\Gamma(a, b, c, d) \subset \text{SPR0}^*$, then $C_{PD}(\alpha(s))$ and $C_{PI}(\alpha(s))$ are strictly passive controllers. Obviously, they are also a dual version of this result.

When the substitution is $\gamma(s) = \frac{as}{cs+d}$ we then have the following interesting result:

Corollary 10

1. *If $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$ robustly stabilizes the family $H_{\gamma^{-1}}(s)$, then the PD controllers:*

$$\widehat{C}_{PD}(s) = K_p + \frac{\widehat{K}_D s}{s + q}$$

with $\widehat{K}_D = \frac{aK_D}{a+rc}$ and $q = \frac{rd}{a+rc}$, stabilizes $H(s)$, for any real a, b, c, d such that $a, c, d > 0$ and $b = 0$;

2. *If $C_{PI}(s) = K_p + \frac{K_I}{s}$ robustly stabilizes the family $H_{\gamma^{-1}}(s)$, then the PI controllers:*

$$\widehat{C}_{PI}(s) = \widehat{K}_p + \frac{\widehat{K}_I}{s}$$

with $\widehat{K}_p = K_p + \frac{K_I c}{a}$ and $\widehat{K}_I = \frac{dK_I}{a}$, stabilizes $H(s)$, for any real a, b, c, d such that $a, c, d > 0$ and $b = 0$;

3. *If $C_{PID}(s) = \left(K_p + \frac{K_I}{s}\right)\left(K_p + \frac{K_D s}{s+r}\right)$ robustly stabilizes the family $H_{\gamma^{-1}}(s)$, then the PID controllers:*

$$\widehat{C}_{PID}(s) = \left(\widehat{K}_p + \frac{\widehat{K}_I}{s}\right)\left(K_p + \frac{\widehat{K}_D s}{s+q}\right)$$

with $\widehat{K}_p = K_p + \frac{K_I c}{a}$, $\widehat{K}_I = \frac{dK_I}{a}$ and $\widehat{K}_D = \frac{aK_D}{a+rc}$, stabilizes $H(s)$, for any real a, b, c, d such that $a, c, d > 0$ and $b = 0$.

Proof: The proof follows directly from Lemma 5 and Proposition 8. □

Clearly, the substitution $\gamma(s)$, preserves the structure of the PD and PI controllers. In the case of PD controllers it is interesting to note that the K_p constant doesn't change. This can be interpreted in the following way: the predictive part of the PD controller can be modified following the relations:

$$\widehat{K}_D = \frac{aK_D}{a + rc}, \quad q = \frac{rd}{a + rc}$$

They can be seen as a pseudo-parametrization of the derivative part. We can then used this information to develop in the future tuning rulers for the derivative part of the controller. In the same way, we can see that in the case of PI controllers the gains change following

$$\widehat{K}_p = K_p + \frac{K_I c}{a}, \quad \widehat{K}_I = \frac{dK_I}{a}$$

By using standard results on passivity, we can to give the following result.

Corollary 11 *Consider the following controllers:*

1. *$C_1(s) = C_{PI}(s) = K_p + \frac{K_I}{s}$ where $K_p, K_I > 0$.*

2. *$C_2(s) = C_{PD}(s) = K_p + \frac{K_D s}{s+r}$ where $r, K_p, K_D > 0$.*

3. *$C_3(s) = C_{LL}(s) = K_p \frac{1+T_N s}{1+T_D s}$ where $K_p, T_D, T_N > 0$.*

4. *$C_4(s) = C_{PID_1}(s) = K_p + \frac{K_I}{s} + \frac{K_D s}{s+r}$ where $r, K_p, K_I, K_D > 0$.*

5. *$C_5(s) = C_{PID_2}(s) = K_p + \frac{K_I}{s} + K_D s$ where $K_p, K_I, K_D > 0$.*

6. *$C_6(s) = C_{PID_3}(s) = K_p \left(\frac{1+T_i s}{T_i s}\right) \frac{1+T_d s}{1+\eta T_d s}$ where $K_p > 0$, $0 < T_d < T_i$ and $0 < \eta \leq 1$.*

7. *$C_7(s) = C_{PID_4}(s) = K_p \beta \left(\frac{1+T_i s}{1+\beta T_i s}\right) \frac{1+T_d s}{1+\eta T_d s}$ where $K_p > 0$, $0 < T_d < T_i$, $1 \leq \beta$ and $0 < \eta \leq 1$.*

Now the following assumption is made: Given a fixed plant $H(s)$, there exists a subset Ω of linear transformations $\alpha(s) = \frac{as+b}{cs+d}$ where a, b, c, d are real numbers, such that $H(\alpha(s))$ is a PR *function for each $\alpha(s) \in \Omega$.*

Then, for all SPR0 *function $\nu(s)$ and for all $\alpha(s) \in \Omega$, the controller $C_j(\nu(s))$ stabilizes the plant $H(\alpha(s))$ for $j = 1, \ldots, 7$.*

Proof: It is sufficient to observe that the controllers $C_j(\nu(s))$ are strict input passive for $j = 1, \ldots, 7$, if $\nu(s)$ is an SPR0 function, and that $H(\alpha(s))$ is PR function for each $\alpha(s) \in \Omega$. Hence, using the standard stabilization result on feedback of passive systems and strict input passive systems, the corollary is proven (see Theorem 3.5 in [9]). □

Notice that the plant $H(s)$ can be unstable and non minimum phase and that the controller $C_7(\nu(s))$ stabilizes the plant $H(\alpha(s))$ for any $\nu(s) \in$ SPR0*.

Corollary 12 *Let us consider the plant $H(s)$ and one of the following controllers $C_{PD}(s)$, $C_{PI}(s)$ or $C_{PID}(s)$ stabilizes the plant. Also consider the linear transformations $\gamma_1(s) = \frac{as}{cs+d}$ or $\gamma_2(s) = \frac{as+b}{cs}$ were a, b, c, d are positive real numbers, and let us substitute the s variable by $\gamma_1^{-1}(s) = \frac{ds}{a-cs}$ or $\gamma_2^{-1}(s) = \frac{b}{cs-a}$ in $H(s)$. Then:*

1. *The controllers of the form $C_{\gamma_k}(s) \equiv C(\gamma_k(s))$ stabilizes $H(s)$, if $C(s)$ stabilizes in a robust way the plant $H_{\gamma_k^{-1}}(s) \equiv H(\gamma_k^{-1}(s))$ for $k = 1, 2$. Where $C_{\gamma_k}(s)$ is one of the following controllers $C_{PD}(\gamma_k(s))$, $C_{PI}(\gamma_k(s))$ or $C_{PID}(\gamma_k(s))$.*

2. *The controller $C(s)$ stabilizes in a robust way the plants $H_{\gamma_k}(s) \equiv H(\gamma_k(s))$, if the controllers $C(\gamma_k^{-1}(s))$ stabilizes the plant $H(s)$ for $k = 1, 2$. Where $C(\gamma_k^{-1}(s))$ is one of the following controllers $C_{PD}(\gamma_k^{-1}(s))$, $C_{PI}(\gamma_k^{-1}(s))$ or $C_{PID}(\gamma_k^{-1}(s))$.*

Proof: The proof is consequence of the Proposition 8. □

Corollary 12 can be interpreted as a dual result, in the same sense that the Proposition 8. Moreover, in this case by Corollary 10, notice that $C_{PD}(\gamma_1(s))$ and $C_{PD}(\gamma_1^{-1}(s))$ are PD controllers, $C_{PI}(\gamma_1(s))$ and $C_{PI}(\gamma_1^{-1}(s))$ are PI controllers and $C_{PID}(\gamma_1(s))$ and $C_{PID}(\gamma_1^{-1}(s))$ are PID controllers, with new parameters.

Corollary 13 *Consider the controllers $C_i(s)$ for $i = 1, \ldots, 7$. Then:*

1. *For all PR0 function $\sigma(s)$, the controllers $C_j(s)$ are PR0 functions for $j = 1, \ldots, 7$. In particular, for $\gamma_1(s) = \frac{as}{cs+d}$ and $\gamma_2(s) = \frac{as+b}{cs}$ were a, b, c, d are positive real numbers, we have the following:*
   - *(a)* $C_1(\gamma_1(s)) = K_p + \frac{cK_I}{a} + \frac{dK_I}{as}$ *and* $C_1(\gamma_2(s)) = K_p + \frac{K_I cs}{as+b}$ *where* $K_p, K_I > 0$.
   - *(b)* $C_2(\gamma_1(s)) = \frac{(aK_p + K_p rc + K_I a)s + K_p rd}{(a+rc)s+rd}$ *and* $C_2(\gamma_2(s)) = \frac{(aK_p + K_p rc + K_I a)s + K_p b}{(a+rc)s+b}$ *where* $r, K_p, K_D > 0$.
   - *(c)* $C_3(\gamma_1(s)) = K\frac{(c+T_N a)s+d}{(c+T_D a)s+d}$ *and* $C_3(\gamma_2(s)) = K_p\frac{(c+T_N a)s+T_N b}{(c+T_D a)s+T_D b}$ *where* $K_p, T_D, T_N > 0$.
   - *(d)* $C_4(\gamma_1(s)) = K_p + \frac{cK_I}{a} + \frac{dK_I}{as} + \frac{K_D as}{(a+rc)s+rd}$ *and* $C_4(\gamma_2(s)) = K_p + \frac{K_I cs}{as+b} + \frac{K_D as+b}{(a+rc)s+b}$ *where* $r, K_p, K_I, K_D > 0$.
   - *(e)* $C_5(\gamma_1(s)) = K_p + \frac{cK_I}{a} + \frac{dK_I}{as} + \frac{K_D as}{cs+d}$ *and* $C_5(\gamma_2(s)) = K_p + \frac{K_D a}{c} + \frac{K_D b}{cs} + \frac{K_I cs}{cs+d}$ *where* $K_p, K_I, K_D > 0$.
   - *(f)* $C_6(\gamma_1(s)) = K_p\frac{((c+T_d a)s+d)((c+T_i a)s+d)}{((c+\eta T_d a)s+d)T_i as}$ *and* $C_6(\gamma_2(s)) = K_p\frac{((c+T_i a)s+T_i b)((c+T_d a)s+T_d b)}{T_i(as+b)((c+\eta T_d a)s+\eta T_d b)}$ *where* $K_p > 0$, $0 < T_d < T_i$ *and* $0 < \eta \leq 1$.
   - *(g)* $C_7(\gamma_1(s)) = K_p\beta\frac{((c+T_d a)s+d)((c+T_i a)s+d)}{((c+\eta T_d a)s+d)((c+\beta T_i a)s+d)}$ *and* $C_7(\gamma_2(s)) = K_p\beta\frac{((c+T_d a)s+T_d b)((c+T_i a)s+T_i b)}{((c+\eta T_d a)s+\eta T_d b)((c+\beta T_i a)s+\beta T_i b)}$ *where* $K_p > 0$, $0 < T_d < T_i$, $1 \leq \beta$ *and* $0 < \eta \leq 1$.

2. *If given a fixed plant $H(s)$, there exists a subset Ω of linear transformations $\alpha(s) = \frac{as+b}{cs+d}$ were a, b, c, d are positive real numbers, such that $H(\alpha(s))$ is PR function for each $\alpha(s) \in \Omega$. Then for all $\alpha(s) \in \Omega$,*
   - *(a) the controller $C_1(\gamma_2(s))$ stabilizes the plant $H(\alpha(s))$.*
   - *(b) the controllers $C_2(\gamma_1(s))$ and $C_2(\gamma_2(s))$ stabilize the plant $H(\alpha(s))$.*
   - *(c) the controllers $C_3(\gamma_1(s))$ and $C_3(\gamma_2(s))$ stabilize the plant $H(\alpha(s))$, if $(T_N - T_D) \neq 0$.*
   - *(d) the controller $C_4(\gamma_2(s))$ stabilizes the plant $H(\alpha(s))$, if $(K_D - 1)\, a - rc \neq 0$.*
   - *(e) the controller $C_6(\gamma_2(s))$ stabilizes the plant $H(\alpha(s))$, if $K_p > 0$, $0 < T_d < T_i$ and $0 < \eta \leq 1$.*
   - *(f) the controllers $C_7(\gamma_1(s))$ and $C_7(\gamma_2(s))$ stabilize the plant $H(\alpha(s))$, if $0 < T_d < T_i$, $1 \leq \beta$ and $0 < \eta \leq 1$.*

Proof: It is consequence of the Proposition 7, the well-known fact of that composition of PR functions is a PR function, the fact that in general the linear transform $\frac{as+b}{cs+d}$ is SPR0 if $a, b, c, d > 0$ and $ad - bc \neq 0$, and the fact that the controllers in item 2 are SPR0 due to the substitution $s \to \gamma_1(s)$ or $\gamma_2(s)$, then by the Theorem 3.5 in [9]. □

Corollary 13 is a generalization of Corollaries 10 and 11.

4 Example

We take a plant of the form $p_1(s) = \frac{2(s+1)}{s^2+2s-3}$ and a lead-lag controller $c_1(s) = \frac{34.745(s+1.6373)}{s+37.9063}$, which stabilizes this plant. Let $\alpha^{-1}(s) = \frac{b-ds}{cs-a}$, then the closed-loop transfer function is given by

$$H(s) = \frac{c_1(s)p_1(\alpha^{-1}(s))}{1 + c_1(s)p_1(\alpha^{-1}(s))}$$

with denominator $f(s, a, b, c, d) = A_3 s^3 + A_2 s^2 + A_1 s + A_0$ and coefficients

$$\begin{aligned}
A_3 &= -66.49c^2 - 1.0d^2 + 71.49cd \\
A_2 &= 2.0bd + 189.59cd - 71.49bc + 132.98ac \\
&\quad - 5.7077 \times 10^{-2}c^2 - 37.906d^2 - 71.49ad \\
A_1 &= 0.11415ac - 189.59ad + 75.813bd - 66.49a^2 \\
&\quad - 189.59bc - 1.0b^2 + 71.49ab \\
A_0 &= 189.59ab - 37.906b^2 - 5.7077 \times 10^{-2}a^2
\end{aligned}$$

This polynomial is stable if and only if the following inequalities are met: $A_0 > 0$, $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ and $A_1A_2 - A_0A_3 > 0$. Moreover, we require to meet at least one of the conditions 1, 2, 3, 6, or 7, in Lemma 5. Now by item 1 in Proposition 8, the controllers

$$c_1(\alpha(s)) = \frac{1.73\left[\left(10^4 a + 16373c\right)s + 10^4 b + 16373d\right]}{(500a + 18953c)\,s + 500b + 18953d}$$

stabilize the plant $p_1(s)$ for the set of parameters a, b, c, d that met with the last conditions. For example with $a, d \in \left[10.0^{-3}, 8\right]$, $b, c \in [0, 5]$, we get the controllers $c_1(\alpha(s))$ stabilizes $p_1(s)$.

5 Conclusions

We have characterized the space of parameters a, b, c, d for which the map $s \to \alpha(s)$, preserves stability for any stable polynomial, mapping stable polynomials in stable polynomials. But also we characterized the space of parameters a, b, c, d for which the map before mentioned, mapping unstable polynomials in stable polynomials. Like a consequence, we obtain a dual result, in the sense that the robust stabilization of a plant $H(s)$ with disturbances nonlinear in its parameters, induced by the substitution of the variable s by $\alpha^{-1}(s)$, with a controller $C(s)$, implies the nonfragility of the controller $C(s)$ under the same class of disturbances, induced by the substitution of the variable s by $\alpha(s)$, in the controller, and the nonfragility of the controller $C(s)$ under disturbances, induced by the substitution of the variable s by $\alpha^{-1}(s)$, in the controller, implies the robust stabilization of a plant $H(s)$ with disturbances nonlinear in its parameters, induced by the substitution of the variable s by $\alpha(s)$ with a controller $C(s)$. In the particular case when $b = 0$, the substitution $\alpha(s)$, preserves the structure of the controllers type PD/PI/PID and we give some ideas to use later for tuning rules for the derivative part of the controller type PD, and for the proportional and integral part of the controller type PI. Based on the resulting pseudo-parametrization for this type of controller, after of the substitution of the variable s by $\alpha(s)$, taking $b = 0$. Finally, we given results about stabilization based in passivity using the substitutions $\gamma_1(s)$ and $\gamma_2(s)$.

References:

[1] G. Fernández-Anaya, S. Muñoz-Gutiérrez, R. A. Sánchez-Guzmán, and W. W. Mayol-Cuevas, Simultaneous stabilization using evolutionary strategies,*International Journal of Control*, 68 (6), 1417-1435, 1997.

[2] G. Fernández, Preservation of SPR functions and stabilization by substitutions in SISO plants, *IEEE Trans. Automat. Contr.*, 44, 2161-2164, 1999.

[3] G. Fernández-Anaya and J. A. Torres-Muñoz, Preservation of stability in multidimensional systems using SPR functions, *IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications*, 49, 1654-1658, 2002.

[4] G. Fernández-Anaya and V. L. Kharitonov, Powers of SPR Functions and Preservation Properties, *Journal of the Franklin Institute*, 339, 521-528, 2002.

[5] T. E. Djaferis, Stability preserving maps and robust design, *International Journal of Control*, 75, 680-690, 2002.

[6] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, New Jersey, Prentice Hall, 1979.

[7] K. S. Narendra and J. H. Taylor, Frequency Domain Criteria for Absolute Stability, New York, Academic Press, 1963.

[8] B. T. Polyak and Ya. Z. Tsypkin, Stability and robust stability of uniform systems, *Automation and Remote Contr*. 46 (1995) 1505-1516.

[9] R. Lozano, B. Brogliato, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*, Springer-Verlag, London, UK, 2000.

A Appendix

In this appendix, we present the proofs for Lemma 3, Lemma 5 and Lemma 7.

Proof of the Lemma 3. The proof done in two parts based on the factorization $H(s) = H_r(s)H_c(s)$ described in the Preliminaries. The first part shows the stability for the $H_r(s)$ factor that contains only real poles and zeros. The second part shows the stability for $H_c(s)$ factor that contains only complex poles and zeros. The factor $H_r(\alpha(s))$ can be written as

$$H_r(\alpha(s)) = k\beta\,(cs + d)^{s_0}\,\frac{(s - \delta_1)\cdots(s - \delta_{n-l_1})}{(s - \eta_1)\cdots(s - \eta_{m-j_1})}$$

where $\beta^{-1} = \frac{\prod_{i=1}^{m-j_1}(a - p_i c)}{\prod_{i=1}^{n-l_1}(a - z_i c)}$, $s_0 = m - j_1 - (n - l_1)$, $\delta_i = \frac{dz_i - b}{a - cz_i}$ $(i = 1, ..., n - l_1)$; $\eta_i = \frac{dp_i - b}{a - cp_i}$ $(i = 1, ..., m - j_1)$. And the factor $H_c(\alpha(s))$ can be written as

$$H_c(\alpha)) = K(s)\frac{\prod_{i=1}^{l_0} \mathfrak{B}_{2,i}s^2 + \mathfrak{B}_{1,i}s + \mathfrak{B}_{0,i}}{\prod_{i=1}^{j_0} \mathfrak{A}_{2,i}s^2 + \mathfrak{A}_{1,i}s + \mathfrak{A}_{0,i}}$$

with $K(s) = (cs + d)^{j_1 - l_1}$, $\mathfrak{B}_{0,i} = (b - \rho_i d)^2 + \nu_i^2 d^2$, $\mathfrak{B}_{1,i} = 2\left[(a - \rho_i c)(b - \rho_i d) + \nu_i^2 cd\right]$, $\mathfrak{B}_{2,i} = (a - \rho_i c)^2 + \nu_i^2 c^2$, $\mathfrak{A}_{0,i} = (b - \sigma_i d)^2 + \omega_i^2 d^2$, $\mathfrak{A}_{1,i} =$

$2\left[(a-\sigma_i c)(b-\sigma_i d)+\omega_i^2 cd\right]$, $\mathfrak{A}_{2,i}=(a-\sigma_i c)^2+\omega_i^2 c^2$. Now, we assume that $cd \neq 0$ and $ad-bc \neq 0$. In consequence $H_r(\alpha(s))$ is stable i.e., $\eta_i<0$ for $i=1,\ldots,m-j_1$ if and only if either $p_i d-b>0$ and $a-p_i c<0$, or $p_i d-b<0$ and $a-p_i c>0$ for each $i=1,\ldots,m-j_1$. Note that $H_c(\alpha(s))$ is stable if and only if $(a-\sigma_i c)(b-\sigma_i d)+\omega_i^2 cd>0$ for $i=1,\ldots,j_0$, since $cd \neq 0$ this condition is equivalent to $\sigma_i^2-\left(\frac{a}{c}+\frac{b}{d}\right)\sigma_i+\frac{ab}{cd}+\omega_i^2>0$ for $i=1,\ldots,j_0$. Therefore, with these conditions $H(\alpha(s))$ is stable. □

Let $\mathcal{I}=\{1,...,m-j_1\}$ and $\mathcal{J}=\{1,...,j_0\}$.

Proof of the Lemma 5

1. This case was proven in [1].

2. If either $b,c,d>0$ and $a=0$, or $b,c,d<0$ and $a=0$. Then $\frac{p_i d-b}{p_i c}>0$ since $p_i<0\ \forall i\in\mathcal{I}$, and $H_r(\alpha(s))$ is unstable. For the complex case $(a-\sigma_j c)(b-\sigma_j d)+cd\omega_j^2=-\sigma_j c(b-\sigma_j d)+cd\omega_j^2>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$, and $H_c(\alpha(s))$ is stable.

3. If either $a,c,d>0$ and $b=0$ or $a,c,d<0$ and $b=0$. Then $\frac{p_i d}{a-p_i c}<0$ since $p_i<0, \forall i\in\mathcal{I}$, and $H_r(\alpha(s))$ is stable. For the complex case $(a-\sigma_j c)(b-\sigma_j d)+cd\omega_j^2=-\sigma_j d(a-\sigma_j c)+cd\omega_j^2>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

4. If $a,b,d>0$, $c=0$, then $\frac{p_i d-b}{a}<0$ since $p_i<0, \forall i\in\mathcal{I}$, and $H_r(\alpha(s))$ is stable. For the complex part, note that if $c=0$ then $(a-\sigma_j c)(b-\sigma_j d)+cd\omega_j^2=cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-ad\sigma_j+ab>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ is stable.

5. It is similar to item 4.

6. If either $a,d>0$, $b=c=0$ or $a,d<0$, $b=c=0$. Then $\frac{p_i d}{a}<0$ since $p_i<0, \forall i\in\mathcal{I}$ and $H_r(\alpha(s))$ is stable, and $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-ad\sigma_j>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

7. It is similar to item 6., for the real part. In the complex case we have $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=bc\sigma_j<0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $b,c>0$, $a=d=0$ or $b,c<0$, $a=d=0$, then $H_c(\alpha(s))$ is stable.

8. If $a,b>0$, $d<0$, $c=0$, then $\frac{p_i d-b}{a}<0$, since $p_i d-b<0, \forall i\in\mathcal{I}$ and $H_r(\alpha(s))$ is stable. For the complex part $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-ad\sigma_j+ab>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $\max\{\sigma_1,\ldots,\sigma_{j_0}\}<\frac{b}{d}$. Then $H_c(\alpha(s))$ is stable.

9. It is similar to item 8. for the real part. In the complex part we have $-ad\sigma_j+ab>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $a,b<0$, $d<0$, $c=0$. Then $H_c(\alpha(s))$ is stable.

10. If $a<0$, $b>0$, $d<0$, $c=0$ and $p_i d-b>0$, $\forall i\in\mathcal{I}$ then $\frac{p_i d-b}{a}<0$ since $p_i<0, \forall i\in\mathcal{I}$ and $H_r(\alpha(s))$ is stable. If $-d\sigma_j+b>0, \forall j\in\mathcal{J}$, then $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-ad\sigma_j+ab>0$, since $\sigma_j<0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

11. It is similar to item 10.

12. If $a,b<0$, $c<0$, $d=0$, $\frac{-b}{a-p_i c}<0$ since $p_i<0, \forall i\in\mathcal{I}$ and $H_r(\alpha(s))$ is stable. In the complex part we have $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-bc\sigma_j+ab>0$ since $\sigma_j<0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

13. If $a>0$, $b<0$, $c<0$, $d=0$ and $a-p_i c<0$, $\forall i\in\mathcal{I}$, then $\frac{-b}{a-p_i c}<0$, since $p_i<0, \forall i\in\mathcal{I}$ and $H_r(\alpha(s))$ is stable. In the complex part we have $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-bc\sigma_j+ab>0$ since $\sigma_j<0$ and $a-c\sigma_j>0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

□

Proof of the Lemma 7.

1. We assume that $p_i>0, \forall i\in\mathcal{I}$ and $\sigma_j>0, \forall j\in\mathcal{J}$. In this case if $a>0$, $b<0$, $d<0$, $c=0$ and $p_i d-b<0$, then $\frac{p_i d-b}{a}<0$ and $H_r(\alpha(s))$ is stable. If $b-d\sigma_j>0$ then $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-ad\sigma_j+ab>0$, $\forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

2. We assume that $p_i>0, \forall i\in\mathcal{I}$ and $\sigma_j>0, \forall j\in\mathcal{J}$. If $a<0$, $b>0$, $c<0$, $d=0$, and $a-p_i c<0$ then $\frac{-b}{a-p_i c}<0$, and $H_r(\alpha(s))$ is stable. In the complex part we have $cd(\sigma_j^2+\omega_j^2)-(ad+bc)\sigma_j+ab=-bc\sigma_j+ab>0$ since $\sigma_j<0$ and $a-c\sigma_j>0, \forall j\in\mathcal{J}$ and $H_c(\alpha(s))$ is stable.

□