Revista Mexicana de Astronomía y Astrofísica, **41**, 131–135 (2005)

A SIMPLE MODEL FOR HYDROMAGNETIC INSTABILITIES IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD

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Received 2004 March 15; accepted 2005 January 31

RESUMEN

Se deduce la ecuación para la inestabilidad de Jeans en el caso de un plasma diluido completamente ionizado bajo la presencia de un campo gravitacional y otro campo magnético constante. Se obtiene un nuevo número de Jeans substancialmente modificado por la presencia del campo magnético bajo el supuesto de una simetría apropiada. No obstante, nuestro conocimiento actual respecto a la existencia de campos magnéticos primitivos en el universo es tal que, aparentemente, indica que son un tanto irrelevantes para la formación de estructuras en gran escala. Se presenta una estimación de la masa de Jeans considerando campos magnéticos cosmológicos.

ABSTRACT

In this paper we study a simple model consisting of a dilute fully ionized plasma in the presence of the gravitational and a constant magnetic field, so as to analyze the propagation of hydromagnetic instabilities. In particular we show that the so called Jeans instability is in principle affected by the presence of the magnetic field. A brief discussion is made attempting to assess this influence during the stage of the evolution of the Universe when structures were formed. The most logical conclusion is that if magnetic fields existed in those times then their magnitudes were too small to modify the Jeans mass. Our results place limits on the possible values of seed magnetic fields consistent with the formation of structures in the Universe. These values are within the range of the results obtained by other authors.

Key Words: **HYDRODYNAMICS — MAGNETIC FIELDS — MHD — INSTABILITIES**

1. INTRODUCTION

Magnetic fields have a significant effect on virtually all astrophysical objects. They are observed on all scales. Close to home, the Earth has a bipolar magnetic field with a strength of 0.3G at the equator and 0.6G at the poles (Carilli & Taylor 2002). Within the interstellar medium, magnetic fields are thought to regulate star formation via the ambipolar diffusion mechanism (Spitzer 1978). Our Galaxy has a typical interstellar magnetic field strength of $\sim 2\mu$G in both regular ordered and random components. Other spiral galaxies have been estimated to have magnetic field strengths of 5 to 10μG, with fields strengths up to 50μG found in starburst galaxy nuclei (Beck et al. 1996). Also, magnetic fields are fundamental to the observed properties of jets and lobes in radio galaxies, and they may be primary elements in the generation of relativistic outflows from accreting massive black holes (Carilli & Taylor 2002).

Magnetic fields with typical strengths of order 1μG have been measured in the intercluster medium using a variety of techniques. Large variations in the field strength and topology are expected from cluster to cluster, specially when comparing dynamically relaxed clusters to those that have recently undergone a merger. Magnetic fields with strengths of $10 - 40\mu$G have been observed in some locations (Carilli & Taylor 2002). In all cases, the magnetic fields play an important role in the energy transport in the intercluster medium and in gas collapse.

On the other hand at the cosmological level the presence or existence of magnetic fields is more controversial. In a recent review on the subject (Widrow

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2002) it is firmly asserted that a true cosmological magnetic field is one that cannot be associated with collapsing or virialized structures. Thus the particular role that they may play in the epoch of galaxy formation is rather obscure. Although limits have been placed on the strength of cosmological magnetics fields from Faraday rotation studies of high redshift sources, anisotropy measurements of the CMB and the light element abundances from nucleosynthesis, the question remains: Is there the possibility that the Jeans mass arising from gravitational instabilities responsible for galaxy formation be modified by the presence of a magnetic field?

In spite of the dubious background provided by our present knowledge, this question has been tackled since over fifty years ago. In fact, already Chandrasekhar & Fermi (1953) reached the conclusion that the Jeans criterion for the onset of a hydrodynamic instability is unaffected by a magnetic field in an extended homogeneous gas of infinite conductivity in the presence of a uniform magnetic field. However, in their calculation they assumed that within the gas there existed a fluctuating magnetic field. This problem has been studied by several other authors in different contexts. In particular Lou (1996) studied the problem of gravitational collapse in a magnetized dynamic plasma in the presence of a finite amplitude circularly polarized Alfvén wave. This author does find a case in which the Jeans wave number k_J is indeed modified by the magnetic field by a term proportional to $[c_0^2 + c_A^2]^{1/2}$, where c_0 is the velocity of sound and $c_A = Bz_0(4\pi\rho_0)^{-1/2}$ is Alfvén's wave speed, Bz_0 being the z-component of the uniform magnetic field. Other attempts to show that magnetic fields do play an essential role in galaxy formation have been performed, e.g., Kim, Olinto, & Rosner (1996), although not specifically addressing the question of a magnetic instability. Tsagas & Maartens (2000) have performed a magnetohydrodynamical analysis within a relativistic framework, addressing the Jeans instability on the basis of their previous work (Tsagas & Barrow 1997, 1998).

In view of all these efforts we still feel that the simple question of whether or not a dilute non-magnetized plasma cloud placed in the presence of an external, uniform magnetic field in which density fluctuations are also present due to a fluctuating gravitational field, exhibits a Jeans wave number which is modified by the presence of the field, has not yet been fully discussed in the literature. This is the purpose of the present work. The basic and rather simple formalism is given in §2. Section 3 is devoted to the derivation of the dispersion relation leading to the modified form of k_J and some attempts are made to place the relevance of the results within a realistic frame for existing magnetic field intensities. Some concluding remarks are given in §4.

2. BASIC FORMALISM

We start by assuming that the dynamics of the dilute plasma is governed by Euler's equations of hydrodynamics, namely the balance equations for the fluid's mass density $\rho(\vec{r},t)$ and its velocity $u(\vec{r},t)$. Thus,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \,, \tag{1}$$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) + \nabla p = \vec{f}_g + \vec{f}_M \,. \tag{2}$$

In Eq. (2), $\vec{f}_g$ is the force arising from the gravitational field and we assume that the plasma is diluted enough so that due to the enormous mass difference between the ions and the electrons, the effect of the external field $\vec{B}$ will be substantially larger on the former. Thus the Lorentz force $\vec{f}_M = \frac{q}{m}\rho_0(\vec{u}\times\vec{B})$ where m is the mass of the ions having charge q.

Eqs. (1–2) can be linearized by introducing density and velocity fluctuations defined by:

$$\rho = \rho_0 + \delta\rho \,, \tag{3}$$

$$\vec{u} = \vec{u}_0 + \delta\vec{u} \,, \tag{4}$$

and,

$$\delta\theta \equiv \nabla \cdot (\delta\vec{u}) \;, \tag{5}$$

where ρ_0 is the average density. The fluid is assumed to be static, so that $\vec{u}_0 = 0$; φ represents the gravitational potential and the external magnetic force is that corresponding to a constant magnetic field $\vec{B} = (B_0 + \delta B)\,\hat{k}$, so that the linearized equations for the density and velocity fluctuations can be written as:

$$\frac{\partial (\delta\rho)}{\partial t} + \rho_0\delta \,, \theta = 0 \,, \tag{6}$$

and

$$\rho_0 \frac{\partial (\delta\vec{u})}{\partial t} + \nabla(\delta p) = -\rho_0 \nabla(\delta\varphi) + \frac{q}{m}\rho_0 \left(\delta\vec{u}\times\vec{B}_0\right) \,, \tag{7}$$

where $\vec{f}_g = -\nabla(\delta\varphi)$, $\delta\varphi$ being the fluctuating gravitational field. Neglecting temperature fluctuations, the pressure term in Eq. (7) may be rewritten in terms of the density fluctuations through the local equilibrium assumption, namely, $p = p(\rho)$ so that

$$\nabla p = \left(\frac{\partial p}{\partial \rho_0}\right)_T \nabla\rho = \frac{c_0^2}{\gamma}\nabla\rho.$$

We know recall that K_T, the thermal compressibility satisfies the relation $K_T = \rho\gamma/c_0^2$ where $\gamma = C_p/C_v$ and c_0 is the velocity of sound in the medium. Therefore, Eq. (7) may be rewritten as

$$\rho_0 \frac{\partial\,(\delta\vec{u})}{\partial t} + \frac{c_0^2}{\gamma}\nabla\,(\delta\rho) = -\rho_0\nabla\,(\delta\varphi) + \frac{q}{m}\rho_0\left(\delta\vec{u}\times\vec{B}_0\right). \quad (8)$$

Assuming now that $\delta\varphi$ is defined through Poisson's equation so that $\nabla^2(\delta\varphi) = 4\pi G\delta\rho$ where G is the gravitational constant, that $\vec{B}_o = B_o\hat{k}$ where $\hat{k}$ is the unit vector along the z-axis and noticing that for this case the last term equals $B_o(\hat{i}\delta u_y - \hat{j}\delta u_x)$, Eq. (8) reduces to

$$-\rho_0 \frac{\partial\,(\delta\theta)}{\partial t} + \frac{c_0^2}{\gamma}\nabla^2\,(\delta\rho) = -4\pi G\rho_0\,(\delta\rho) + \frac{q}{m}B_0\rho_0\,(\nabla\times\delta\vec{u})_{\hat{k}}\,, \quad (9)$$

after taking its divergence and using Eq. (6).

Eqs. (6) and (9) are now two simultaneous equations for $\delta\rho$ and $\delta\vec{u}$ which need to be solved. To do so we notice first that $(\nabla\times\delta\vec{u})_{\hat{k}} = -\hat{k}\left[\nabla(\delta\vec{u}) + \frac{\partial(\delta u_z)}{\partial z}\right]$ so that using Eq. (6) we may write

$$\frac{q}{m}B_0\rho_0\frac{\partial}{\partial t}\,(\nabla\times\delta\vec{u})_{\hat{k}} = -\frac{q}{m}B_0\rho_0\left[\frac{\partial}{\partial t}\left(-\frac{1}{\rho_0}\frac{\partial\rho}{\partial t}\right) + \frac{\partial}{\partial t}\frac{\partial(\delta u_z)}{\partial z}\right]. \quad (10)$$

Finally, taking the time derivate of Eq. (9) and using Eqs. (6) and (10) one is led to the result

$$-\frac{\partial^3}{\partial t^3}(\delta\rho) + \frac{c_0^2}{\gamma}\nabla^2\left(\frac{\partial(\delta\rho)}{\partial t}\right) + 4\pi G\rho_0\frac{\partial}{\partial t}(\delta\rho) - \left(\frac{qB_0}{m}\right)\frac{\partial}{\partial t}\left(\frac{\partial(\delta\rho)}{\partial t}\right) - \left(\frac{qB_0}{m}\right)\rho_0\frac{\partial}{\partial t}\left(\frac{\partial(\delta u_z)}{\partial z}\right) = 0\,. \quad (11)$$

Integrating once with respect to time and setting the integration constant equal to zero, (which does not affect the validity to our argument) we get

$$-\frac{\partial^2}{\partial t^2}(\delta\rho) + \frac{c_0^2}{\gamma}\nabla^2(\delta\rho) + 4\pi G\rho_0(\delta\rho) - \left(\frac{qB_0}{m}\right)\left(\frac{\partial(\delta\rho)}{\partial t}\right) - \left(\frac{qB_0}{m}\right)\rho_0\left(\frac{\partial(\delta u_z)}{\partial z}\right) = 0\,. \quad (12)$$

Equation (12) is now a single equation for the density fluctuations $\delta\rho$. Indeed, since B_0 points along the z-axis, $[\delta\vec{u}\times\vec{B}_0]_{\hat{k}} = 0$ so that Eq. (8) reduces to

$$\rho_0\frac{\partial(\delta u_z)}{\partial t} + \left(\frac{c_0^2}{\gamma} - 4\pi G\rho_0\right)\frac{\partial(\delta\rho)}{\partial z} = 0\,. \quad (13)$$

The solution to Eqs. (12) and (13) is readily achieved proposing that $\delta\rho$ is described by a plane wave, namely

$$\delta\rho = Ae^{i(\vec{k}.\vec{r}-\omega t)}\,. \quad (14)$$

Taking the partial derivative with respect to z in Eq. (13), exchanging the time and space derivatives in the first term and calculating $\frac{\partial^2}{\partial z^2}(\delta\rho)$ we get

$$\rho_0\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}(\delta u_z)\right) + \left(\frac{c_0^2}{\gamma} - 4\pi G\rho_0\right)\left(-k_z^2\delta\rho\right) = 0\,. \quad (15)$$

This result implies that

$$\rho_0\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}(\delta u_z)\right) = k_z^2\Gamma Ae^{i(\vec{k}.\vec{r}-\omega t)}\,, \quad (16)$$

where $\Gamma = \frac{c_0^2}{\gamma} - 4\pi G\rho_0$. If we now use Eq. (16) back in Eq. (11) and use Eq. (13) for the density fluctuations, we arrive at the following dispersion relation

$$\omega^3 + i\omega^2\omega_0 - \omega(4\pi G\rho_0 - \frac{k^2c_0^2}{\gamma}) - i\omega_0(\frac{c_0^2}{\gamma} - 4\pi G\rho_0)k_z^2 = 0\,, \quad (17)$$

after multiplication by i.

Notice that in Eq. (17) each term has a clear meaning. The term quadratic in ω contains only the contribution of the magnetic mode, $\omega_0 = qB_0/m$, the term linear in ω arises from the mechanical modes, sound plus gravitation, and the third term is a coupling of both along the z-axis, the one chosen as the direction of the magnetic field. If we now assume a weak coupling limit, namely that the magnetic field has a small effect on the mechanical modes we may, to a first approximation, neglect this term so that Eq. (17) reduces to

$$\omega^2 + i\omega\omega_0 - (4\pi G\rho_0 - \frac{k^2c_0^2}{\gamma}) = 0\,, \quad (18)$$

a simple quadratic equation for ω.

That this approximation is justified in the long time, small frequency limit may be seen from

Eq. (16) itself. Indeed the left hand side of this equation is real so that the right hand side must be proportional to $cos(\omega t)$. Thus on the average, in such a limit $< cos(\omega t) >= 0$ implying that $\partial/\partial_z(u_z) = 0$. But this is precisely what the weak coupling limit states, that the magnetic field induces small fluctuations in the fluid velocity along the z-axis but that the spatial gradient of such fluctuations is very small.

In this approximation we have to find the roots of Eq. (18) and instabilities arise when such roots are imaginary. An elementary analysis of this equation selecting the only root with a non-trivial meaning leads, to a first approximation, to the threshold value of k beyond which this happens

$$k_J^2 = \frac{\gamma}{c_0^2}\left(4\pi G\rho_0 - \frac{1}{2}\left(\frac{qB_0}{m}\right)\right) . \qquad (19)$$

This is precisely the Jeans wave number modified by the presence of a constant magnetic field. Clearly, if $B_0 = 0$ we recover the well-known expression for k_J. Moreover, due to the results to be discussed hereafter, we believe that a more detailed analysis withdrawing this assumption is not necessary. The question now is how relevant the second term is in hindering structure formation. This will be analyzed in the following section.

3. ANALYSIS OF THE DISPERSION RELATION

As indicated in the previous section, the linearized version of fluctuating non-dissipative hydrodynamics predicts for a dilute non-magnetized homogeneous fully ionized plasma in the presence of a uniform magnetic field a Jeans wave number which is, as depicted in Eq. (19), a competition between the gravitational and magnetic fields. The question is whether this result has any bearing on the value of the Jeans mass in realistic cases. Clearly, Eq. (19) points at two possibilities, namely, if the magnetic term is negligible or of the same order as the gravitational one. As we recall, the Jeans mass is defined as

$$M_J \equiv \frac{4\pi}{3}\rho_o\lambda_J^3 = \frac{4\pi}{3}\rho_o\left(\frac{2\pi}{k_J}\right)^3 ,$$

so that using Eq. (19)

$$M_J = \frac{32\pi^4}{3}\rho_o\left[\frac{c_o^2}{\gamma}\frac{1}{4\pi G\rho_o - \frac{1}{2}\left(\frac{qB_o}{m}\right)^2}\right]^{3/2} , \qquad (20)$$

where $\rho_o = mn_o$, n_o being the particle density in the plasma. As it has been exhaustively discussed in the literature (Jeans 1945; de Groot & Mazur 1984; Peebles 1993; Weinberg 1971; Sandoval-Villalbazo & García-Colín 2002) without the magnetic contribution, present values of $\rho_o \sim 10^{-29}$ g/cm^3, $m = m_H \sim 10^{-24}$ g and $T \sim 10^5$ K yield for $M_J \sim 10^{11}$ $M_\odot$.

Nevertheless, a thoughtful examination of Eqs. (19) and (20) is required. Firstly, one should notice that in order for these results be physically meaningful, $4\pi\rho_0 G > 0.5(qB_0/m)^2$ must be fulfilled. This inequality involves two critical parameters namely, ρ_0 and B_0 precisely at the stage where structures are beginning to develop in the evolution of the Universe. Next, since the proton charge-mass ratio is approximately equal to 10^8 c/kg, even for small fields the cyclotron frequency is quite large, unless B_0 is very small, of the order of 10^{-24}G. Thus, we may think of these results as putting a limit on the value of the seed fields that could have existed when structures began to form. In fact, since it is known (Silk 1980) that the average matter density prevailing when our own Galaxy was formed was approximately equal to 10^{-22} kg/m^3, Eqs. (19) and (20) hold only if B_0 is of the order of 10^{-24}G. This is in reasonable agreement with the conclusions reached by several authors that have examined the connection between the creation of the first fields and the formation of large scale structures. Although the values reported seem to depend on the cosmological model (Widrow 2002), several estimates seem to lie in the interval $10^{30}\text{G} < B_0 < 10^{-19}$G. According to our results if ρ_0 is around the value quoted above then, if magnetic fields existed in the cosmos they could have hardly exceeded a value of 10^{-24}G if we take as $M_J \sim 10^{12}$ $M_\odot$. If no seed magnetic field existed then the standard and well established result for the Jeans mass obviously remains unaffected. Notice however, that this parameter could be enhanced by seed magnetic fields for which the denominator in Eq. (20) becomes small but positive. This remains to be tested.

4. CONCLUSIONS

The simple model here discussed for a fully ionized dilute plasma in the presence of a homogeneous uniform magnetic field shows clearly the rather peculiar behavior of the Jeans mass due to the confining effect of the magnetic field. Although such an effect could be important, it would require very high densities and small magnetic fields. The densities required would be so high that the model itself would become dubious, and such values are completely at odds with observations. On the other hand, for low densities the conclusion is that gravitational instabilities will occur in the absence of magnetic fields or, as envisaged by some authors (Widrow 2002) with possible cosmological fields $\sim 10^{-24}$G. Such fields have not yet been detected.

This work was performed under grants from CONACyT 41081-F and 42809.

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